Schauder's estimates for nonlocal equations with singular Lévy measures

Mingyan Wu[1]

Based on a joint work with Zimo Hao[1] and Guohuan Zhao

[1] School of Mathematics and Statistics, Wuhan University

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Plan of this talk

Part 1: Introduction

Lévy measures

Definition 1 (Lévy measures)

ν is a Lévy measure on $\mathbb{R}^d$, if it is a σ-finte (positive) measure such that

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}^d} \left(1 \wedge |x|^2\right) \nu(\mathrm{d}x) < +\infty.$$

Definition 2 (α-stable Lévy measures)

For $\alpha \in (0,2)$, Lévy measure $\nu^{(\alpha)}$ is an α-stable Lévy measure, if it has the polar coordinates form

$$\nu^{(\alpha)}(A) = \int_0^\infty \left(\int_{\mathbb{S}^{d-1}} \frac{1_A(r\theta)\Sigma(\mathrm{d}\theta)}{r^{1+\alpha}} \right) \mathrm{d}r, \quad A \in \mathscr{B}(\mathbb{R}^d),$$

where Σ is a finite measure over the unit sphere $\mathbb{S}^{d-1}$ (called spherical measure of $\nu^{(\alpha)}$).

α-stable Lévy measures

- **Scaling property** $\nu^{(\alpha)}(\mathrm{d}(\lambda z)) = \lambda^{-\alpha}\nu^{(\alpha)}(\mathrm{d}z)$ for any $\lambda > 0$.
- **Moments property** For any $\gamma_1 > \alpha > \gamma_2 \geqslant 0$,

$$\int_{\mathbb{R}^d} (|z|^{\gamma_1} \wedge |z|^{\gamma_2})\nu^{(\alpha)}(\mathrm{d}z) < \infty.$$

Definition 3 (Non-degenerate Lévy measures)

One says that an α-stable Lévy measure $\nu^{(\alpha)}$ is non-degenerate if

$$\int_{\mathbb{S}^{d-1}} |\theta_0 \cdot \theta|^\alpha \Sigma(\mathrm{d}\theta) > 0 \quad \text{for every } \theta_0 \in \mathbb{S}^{d-1}.$$

Non-degenerate α-stable Lévy measures

Example 4 (Standard α-stable Lévy measures)

If Σ is rotationally invariant with $\Sigma(\mathbb{S}^{d-1}) = |\mathbb{S}^{d-1}|$, then $\nu^{(\alpha)}$ is the standard or strict α-stable Lévy measure and

$$\nu^{(\alpha)}(\mathrm{d}y) = \frac{\mathrm{d}y}{|y|^{d+\alpha}}.$$

The d-dimensional Lévy process associated with this Lévy mesaure is called d-dimensional α-stable process.

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- If $W_t = (W_t^1, \cdots, W_t^d)$ is a d-dimensional Browinian Motion, then W_t^i are i.i.d 1-dimensional Browinian Motions.

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- Let $L_t = (L_t^1, \cdots, L_t^d)$ be a d-dimensional α-stable process. L_t^i may not be independet or 1-dimensional standard α-stable processes.

Non-degenerate α-stable Lévy measures

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Question

If $L_t^i, i = 1, \cdots, d$ are i.i.d 1-dimensional standard α-stable processes, then what is $L_t = (L_t^1, \cdots, L_t^d)$?

Non-degenerate α-stable Lévy measures

Example 5 (Cylindrical α-stable Lévy measures)

If $\Sigma = \sum_{k=1}^{d} \delta_{e_k}$, where $e_k = (0, \cdots, 1_{k_{th}}, \cdots, 0)$, then

$$\nu^{(\alpha}(\mathrm{d}x) = \sum_{k=1}^{d} \delta_{e_k}(\mathrm{d}x) \frac{\mathrm{d}x_k}{|x_k|^{\alpha+1}},$$

called cylindrical α-stable Lévy measures. Moreover, this Lévy measure is associated with a d-dimensional Lévy process $(L_t^1, L_t^2, \cdots, L_t^d)$, where $L_t^1, L_t^2, \cdots, L_t^d$ are i.d.d 1-dimensional standard α-stable processes.

Infinitesimal generators

- **Infinitesimal generators**
  - Standard α-stable Lévy process with $\alpha \in (0,1)$:

$$\mathscr{L} f(x) = \mathrm{p.v.} \int_{\mathbb{R}^d} \frac{f(x+z) - f(x)}{|z|^{d+\alpha}} dz = \Delta^{\alpha/2} f(x).$$

  - Cylindrical α-stable Lévy process with $\alpha \in (0,1)$:

$$\mathscr{L} f(x) = \sum_{i=1}^{d} \mathrm{p.v.} \int_{\mathbb{R}} \frac{f(x+ze_i) - f(x)}{|z|^{1+\alpha}} dz,$$

where $e_i = (0, \ldots, 1_{i_{th}}, \ldots, 0)$.

Fourier's multipliers

- **Fourier's multipliers**
  - Standard α-stable Lévy process:
  $$\mathscr{F}(\mathscr{L}f)(\xi) = |\xi|^{\alpha}\mathscr{F}f(\xi) = \mathscr{F}(\Delta^{\frac{\alpha}{2}}f)(\xi),$$
  where $\phi(\xi) := |\xi|^{\alpha} \in C^{\infty}(\mathbb{R}^d \setminus \{0\})$.
  - Cylindrical α-stable Lévy process:
  $$\mathscr{F}(\mathscr{L}f)(\xi) = \sum_{i=1}^{d} |\xi_i|^{\alpha}\mathscr{F}f(\xi),$$
  where $\phi(\xi) := \sum_{i=1}^{d} |\xi_i|^{\alpha} \in C^{\infty}(\mathbb{R}^d \setminus \cup_{i=1}^{d}\{x_i = 0\})$.
- **Note** It is more difficult to deal with cylindrical Lévy measues than standard Lévy measues.

Our work

- We want to show **Schauder's estimates** for the following nonlocal equations:

$$\partial_t u = \mathscr{L}^{(\alpha)}_{\kappa,\sigma} u + b \cdot \nabla u + f, \quad u(0) = 0, \tag{2.1}$$

where $\mathscr{L}^{(\alpha)}_{\kappa,\sigma}$ is an α-stable-like operator with the form:

$$\mathscr{L}^{(\alpha)}_{\kappa,\sigma} u(t,x) := \int_{\mathbb{R}^d} (u(t, x + \sigma(t,x)z) - u(t,x) - \sigma(t,x) z^{(\alpha)} \cdot \nabla u) \kappa(t,x,z) \nu^{(\alpha)}(\mathrm{d}z),$$

where $\nu^{(\alpha)}$ is a non-degenerate α-stabe Lévy measure and $z^{(\alpha)} := z \mathbf{1}_{\{|z|\leqslant 1\}} \mathbf{1}_{\alpha=1} + z \mathbf{1}_{\alpha \in (1,2)}$.

- **Schauder's estimates:**

$$\|u\|_{\mathbf{C}^{\alpha+\beta}} \leqslant c \|f\|_{\mathbf{C}^{\beta}}.$$

PDE Schauder's estimates play a basic role in constructing classical solutions for quasilinear PDEs.

SDE The Schauder estimate can be used to solve the exisitence and uniqueness of the solution for SDE. (The Zvonkin transform)

Supercritical Case: $\alpha \in (0,1)$

- Supercritical case: If $\alpha \in (0,1)$, then

$$\partial_t u = \mathscr{L}^{(\alpha)}_{\kappa,\sigma} u + b \cdot \nabla u + f, \;\; u(0) = 0,$$

with

$$\mathscr{L}^{(\alpha)}_{\kappa,\sigma} u(t,x) := \int_{\mathbb{R}^d} \Big(u(t, x + \sigma(t,x) z) - u(t,x) \Big) \kappa(t,x,z) \nu^{(\alpha)}(\mathrm{d}z).$$

When $\alpha \in (0,1)$, the drift term will play the important role instead of the diffusion term.

Previous results

2009 (Bass) Consider the elliptic equation $\mathscr{L}u = f$, where $\alpha \in (0,2)$ and

$$\mathscr{L}u = \int_{\mathbb{R}^d} (u(x+z) - u(x) - z\mathbf{1}_{\{|z|\leqslant 1\}}\mathbf{1}_{\alpha\in[1,2)} \cdot \nabla u(x)) \frac{\kappa(x,z)}{|z|^{d+\alpha}} \mathrm{d}z.$$

2012 (Silvestrei) Consider the following parabolic equation:

$$\partial_t u + b \cdot \nabla u + (-\Delta)^{\alpha/2} = f,$$

where $\alpha \in (0,1)$ and b is bounded but not necessarily divergence free.

2018 (Zhang, Zhao) Consider

$$\partial_t u = \mathscr{L}u + b \cdot \nabla u + f,$$

where b is bounded globally Hölder function and

$$\mathscr{L}u = \int_{\mathbb{R}^d} (u(x+z) - u(x) - z^{(\alpha)} \cdot \nabla u(x)) \frac{\kappa(x,z)}{|z|^{d+\alpha}} \mathrm{d}z$$

with $\alpha \in (0,2)$ and $z^{(\alpha)} = z\mathbf{1}_{\{|z|\leqslant 1\}}\mathbf{1}_{\alpha=1} + z\mathbf{1}_{\alpha\in(1,2)}$.

2019 (Chaudru de Raynal, Menozzi, Priola) Consider

$$\partial_t u + \mathscr{L}u + b \cdot \nabla u = -f, u(T) = g,$$

where b is unbounded local Hölder function and

$$\mathscr{L}u = \int_{\mathbb{R}^d} (u(x+z) - u(x))\nu(\mathrm{d}z)$$

with singular Lévy measure ν and $\alpha \in (1/2, 1)$.

Assumptions on diffuision coeffients

Recall

$$\partial_t u = \mathscr{L}^{(\alpha)}_{\kappa,\sigma} u + b \cdot \nabla u + f, \quad u(0) = 0, \tag{2.2}$$

where

$$\mathscr{L}^{(\alpha)}_{\kappa,\sigma} u(t,x) := \int_{\mathbb{R}^d} (u(t, x + \sigma(t,x)z) - u(t,x) - \sigma(t,x) z^{(\alpha)} \cdot \nabla u) \kappa(t,x,z) \nu^{(\alpha)}(\mathrm{d}z).$$

($\mathbf{H}^\beta_\kappa$) For some $c_0 \geqslant 1$ and $\beta \in (0,1)$, it holds that for all $x, z \in \mathbb{R}^d$,

$$c_0^{-1} \leqslant \kappa(t,x,z) \leqslant c_0, \quad [\kappa(t,\cdot)]_{\mathbf{C}^\beta} := \sup_h \frac{\|\kappa(t, \cdot + h, z)) - \kappa(t,\cdot,z)\|_\infty}{|h|^\beta} \leqslant c_0,$$

and in the case of $\alpha = 1$,

$$\int_{r \leqslant |z| \leqslant R} z \kappa(t,x,z) \nu^{(\alpha)}(\mathrm{d}z) = 0 \ \text{ for every } 0 < r < R < \infty.$$

($\mathbf{H}^\gamma_\sigma$) For some $c_0 \geqslant 1$ and $\gamma \in (0,1]$, it holds that for all $x, \xi \in \mathbb{R}^d$,

$$c_0^{-1} |\xi|^2 \leqslant \xi^T \sigma(t,x) \xi \leqslant c_0 |\xi|^2, \quad |\sigma(t,x) - \sigma(t,y)| \leqslant c_0 |x - y|^\gamma.$$

Assumptions on drift coeffients

($\mathbf{H}_b^\beta$) For some $c_0 \geqslant 1$ and $\beta \in (0,1)$, it holds that for all $t \in \mathbb{R}$,

$$|b(t,0)| \leqslant c_0, \quad [b(t,\cdot)]_{\mathbb{C}^\beta} := \sup_{0<|h|\leqslant 1} \frac{\|b(t,\cdot+h) - b(t,\cdot)\|_\infty}{|h|^\beta} \leqslant c_0.$$

That is the Local Hölder regularity.

- Here, b is not necessarily bounded in x. For example, $b(x) = x$ satisfies

$$[b]_{\mathbb{C}^s} < \infty, \quad \forall s \in (0,1).$$

  It is related to the nonlocal Ornstein-Uhlenbeck operator:

$$\Delta^{\alpha/2} - x \cdot \nabla.$$

- For any fixed x, function $t \to b(t,x)$ is bounded because

$$\begin{aligned} |b(t,x) - b(t,y)| \leqslant & [b(t,\cdot)]_{\mathbb{C}^\beta} |x-y| \mathbf{1}_{\{|x-y|>1\}} \\ & + [b(t,\cdot)]_{\mathbb{C}^\beta} |x-y|^\beta \mathbf{1}_{\{|x-y|\leqslant 1\}}. \end{aligned} \tag{2.3}$$

Littlewood-Paley Decomposition

- Let ϕ_0 be a radial C^∞-function on $\mathbb{R}^d$ with

$$\phi_0(\xi) = 1 \text{ for } \xi \in B_1 \text{ and } \phi_0(\xi) = 0 \text{ for } \xi \notin B_2.$$

For $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^d$ and $j \in \mathbb{N}$, define

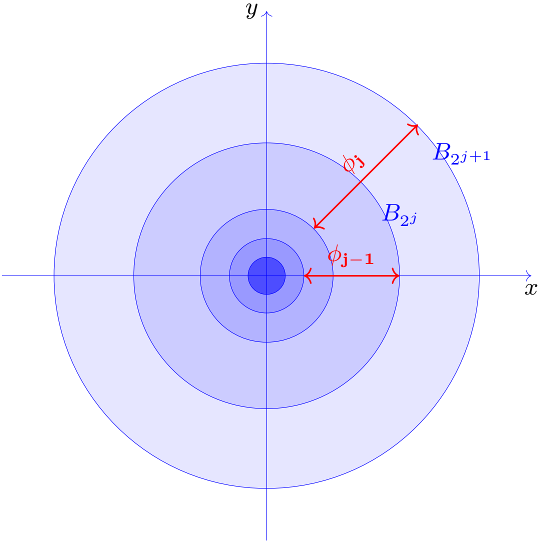
$$\phi_j(\xi) := \phi_0(2^{-j}\xi) - \phi_0(2^{-(j-1)}\xi).$$

- It is easy to see that for $j \in \mathbb{N}$, $\phi_j(\xi) = \phi_1(2^{-(j-1)}\xi) \geqslant 0$ and

$$\text{supp}\phi_j \subset B_{2^{j+1}} \setminus B_{2^{j-1}}, \quad \sum_{j=0}^{k} \phi_j(\xi) = \phi_0(2^{-k}\xi) \to 1, \quad k \to \infty.$$

Littlewood-Paley Decomposition

- If $|j-j'| \geqslant 2$, then $\mathrm{supp}\phi_1(2^{-j}\cdot) \cap \mathrm{supp}\phi_1(2^{-j'}\cdot) = \emptyset$.



Block operators

- For $j \in \mathbb{N}_0$, the block operator Δ_j is defined on $\mathscr{S}'(\mathbb{R}^d)$ by

$$\Delta_j f(x) := (\phi_j \hat{f})^{\check{}}(x) = \check{\phi}_j * f(x) = 2^{j-1} \int_{\mathbb{R}^d} \check{\phi}_1(2^{j-1} y) f(x-y) \mathrm{d}y.$$

- For $j \in \mathbb{N}_0$, by definition it is easy to see that

$$\Delta_j = \Delta_j \widetilde{\Delta}_j, \quad \text{where } \widetilde{\Delta}_j := \Delta_{j-1} + \Delta_j + \Delta_{j+1} \text{ with } \Delta_{-1} \equiv 0,$$

and Δ_j is symmetric in the sense that

$$\langle \Delta_j f, g \rangle = \langle f, \Delta_j g \rangle.$$

- Noticing that

$$\sum_{j=0}^{k} \Delta_j f = 2^{dk} \int_{\mathbb{R}^d} \check{\phi}_0(2^k(x-y)) f(y) \mathrm{d}y \to f, \tag{3.1}$$

we have the Littlewood-Paley decomposition of f:

$$f = \sum_{j=0}^{\infty} \Delta_j f.$$

The Littlewood-Paley characteristic of Hölder spaces

Then, we have the following definition.

Definition 6 (Besov spaces)

For $s \in \mathbb{R}$, the Besov space $\mathbf{B}^s_{\infty,\infty}$ is defined as the set of all $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{\mathbf{B}^s_{\infty,\infty}} := \sup_{j \geqslant -1} 2^{js} \|\Delta_j f\|_\infty < \infty.$$

Proposition 7 (H. Triebel)

For any $0 < s \notin \mathbb{N}_0$, it holds that

$$\|f\|_{\mathbf{B}^s_{\infty,\infty}} \asymp \|f\|_{\mathbf{C}^s},$$

where $\mathbf{C}^s$ is the usual Hölder space. For $n \in \mathbb{N}$, we have $\mathbf{C}^n \subset \mathbf{B}^n_{\infty,\infty}$.

Proposition 8 (Interpolation inequalities)

For any $0 < s < t$, there is a constant $c > 0$ such that for any $\varepsilon \in (0,1)$,

$$\|f\|_{\mathbf{B}^s_{\infty,\infty}} \leqslant \|f\|^{s/t}_{\mathbf{B}^t_{\infty,\infty}} \|f\|^{(t-s)/t}_{\mathbf{B}^0_{\infty,\infty}} \leqslant \varepsilon \|f\|_{\mathbf{B}^t_{\infty,\infty}} + c\varepsilon^{-s/(t-s)} \|f\|_\infty.$$

Part 2: Main results

Classical solutions

Definition 9 (Classical solutions)

We call a bounded continuous function u defined on $\mathbb{R}_+ \times \mathbb{R}^d$ a classical solution of PDE (2.2) if for some $\varepsilon \in (0,1)$,

$$u \in \cap_{T>0} L^\infty([0,T]; \mathbf{C}^{\alpha \vee 1+\varepsilon})$$

with $\nabla u(\cdot, x) \in C([0,\infty))$ for any $x \in \mathbb{R}^d$, and for all $(t,x) \in [0,\infty) \times \mathbb{R}^d$,

$$u(t,x) = \int_0^t \Big(\mathscr{L}^{(\alpha)}_{\kappa,\sigma} u + b \cdot \nabla u + f \Big)(s,x) \mathrm{d}s.$$

Lemma 10 (Maximum principles)

Assume that $\sigma(t,x)$ *and* $\kappa(t,x,z) \geqslant 0$ *are bounded measurable functions. Let* $b(t,x)$ *be measurable function and bounded in* $t \in \mathbb{R}_+$ *for any fixed* $x \in \mathbb{R}^d$*. For any* $T > 0$ *and classical solution* u *of* (2.2) *in the sense of Definitions 9, it holds that*

$$\|u\|_{L^\infty([0,T])} \leqslant T \|f\|_{L^\infty([0,T])}.$$

Schauder's estimates

Theorem 11 (Schauder's estimates)

Suppose that $\gamma \in (0,1]$, $\alpha \in (1/2, 2)$ *and* $\beta \in ((1-\alpha) \vee 0, (\alpha \wedge 1)\gamma)$. *Under conditions* $(\mathbf{H}^{\beta}_{\kappa})$, $(\mathbf{H}^{\gamma}_{\sigma})$, *and* $(\mathbf{H}^{\beta}_{b})$, *for any* $T > 0$, *there is a constant* $c = c(T, c_0, d, \alpha, \beta, \gamma) > 0$ *such that for any classical solution* u *of PDE* (2.2),

$$\|u\|_{L^\infty([0,T],\mathbf{C}^{\alpha+\beta})} \leqslant c\|f\|_{L^\infty([0,T],\mathbf{C}^{\beta})}.$$

- Since we consider classical solutions, $\alpha + \beta$ must be larger than 1 such that ∇u is meaningful. In addition, we must assume $\beta < \alpha$ for the moment problem. Thus, $1 - \alpha < \beta < \alpha$.
- The critical case $\alpha + \beta = 1$ is a technical problem. We have no ideas to fix it.

Existences

Theorem 12 (Existences)

Suppose that $\gamma \in (0,1]$, $\alpha \in (1/2, 2)$ *and* $\beta \in ((1-\alpha)\vee 0, (\alpha \wedge 1)\gamma)$. *Under conditions* $(\mathbf{H}^{\beta}_{\kappa})$, $(\mathbf{H}^{\gamma}_{\sigma})$, *and* $(\mathbf{H}^{\beta}_{b})$, *for any* $f \in \cap_{T>0} L^{\infty}([0,T], \mathbf{C}^{\beta})$, *there is a unique classical solution* u *for* (2.2) *in the sense of Definition 9 such that for any* $T > 0$ *and some constant* $c > 0$,

$$\|u\|_{L^\infty([0,T],\mathbf{C}^{\alpha+\beta})} \leqslant c\|f\|_{L^\infty([0,T],\mathbf{C}^{\beta})}, \quad \|u\|_{L^\infty(0,T)} \leqslant c\|f\|_{L^\infty(0,T)}.$$

Part 3: Sketch of proofs

Main technics

Step 1 Using perturbation argument to prove the Schauder estimate under $(\mathbf{H}^{\beta}_{\kappa})$, $(\mathbf{H}^{\beta}_{\sigma})$ and

$$[b(t,\cdot)]_{\mathbf{C}^{\beta}} \leqslant c_0, \forall t \geqslant 0.$$

- Freeze the coefficients along the characterization curve.
- Use Duhamel's formulas. (Heat kernel estimates of integral form, Littlewood-Paley's decomposition, and interpolation inequalities.)

Step 2 Using cut-off technics to obatin the desired Schauder estimate.

Step 3 By Schauder's estimates, using the continuity method and the vanishing viscosiry approach to get existences of the classical solutions.

The characterization curve

- Fix $x_0 \in \mathbb{R}^d$. Let θ_t solve the following ODE in $\mathbb{R}^d$:

$$\dot{\theta}_t = -b(t, \theta_t), \;\; \theta_0 = x_0.$$

Define

$$\tilde{u}(t,x) := u(t, x+\theta_t), \;\; \tilde{f}(t,x) := f(t, x+\theta_t), \;\; \tilde{\sigma}(t,x) := \sigma(t, x+\theta_t),$$

$$\tilde{\kappa}(t,x,z) := \kappa(t, x+\theta_t, z), \;\; \tilde{\sigma}_0(t) := \tilde{\sigma}(t, 0), \;\; \tilde{\kappa}_0(t,z) := \tilde{\kappa}(t,0,z).$$

and

$$\tilde{b}(t,x) := b(t, x+\theta_t) - b(t, \theta_t).$$

- It is easy to see that $\tilde{u}$ satisfies the following equation:

$$\partial_t \tilde{u} = \mathscr{L}^{(\alpha)}_{\tilde{\kappa}_0, \tilde{\sigma}_0} u + \tilde{b} \cdot \nabla \tilde{u} + \left(\mathscr{L}^{(\alpha)}_{\tilde{\kappa}, \tilde{\sigma}} - \mathscr{L}^{(\alpha)}_{\tilde{\kappa}_0, \tilde{\sigma}_0} \right) \tilde{u} + \tilde{f}.$$

Heat kernels $p_{s,t}^{\tilde{\kappa}_0,\tilde{\sigma}_0}(x)$

- For the case of $\alpha \in (0,1)$. Let $N(\mathrm{d}t, \mathrm{d}z)$ be the Possion random measure with intensity measure

$$\tilde{\kappa}_0(t,z)\nu^{(\alpha)}(\mathrm{d}z)\mathrm{d}t.$$

  For $0 \leqslant s \leqslant t$, define

$$X_{s,t}^{\tilde{\kappa}_0,\tilde{\sigma}_0} := \int_s^t \int_{\mathbb{R}^d} \tilde{\sigma}_0(r) z N(\mathrm{d}r, \mathrm{d}z),$$

  whose generator is $\mathscr{L}^{(\alpha)}_{\tilde{\kappa}_0,\tilde{\sigma}_0}$.

- Under our conditions, the random variable $X_{s,t}^{\tilde{\kappa}_0,\tilde{\sigma}_0}$ has a smooth density $p_{s,t}^{\tilde{\kappa}_0,\tilde{\sigma}_0}(x)$. Moreover, for any $\beta \in [0,\alpha)$ and $m \in \mathbb{N}_0$, there exists a positive constant c such that for all $0 \leqslant s < t$,

$$\int_{\mathbb{R}^d} |x|^\beta |\nabla^m p_{s,t}^{\tilde{\kappa}_0,\tilde{\sigma}_0}(x)| \mathrm{d}x \leqslant c(t-s)^{(\beta-m)/\alpha}, \tag{5.1}$$

  where the constant c depends on $d, m, c_0, \nu^{(\alpha)}, \beta, \alpha$.

Duhamel's formulas

- By Duhamel's formula and operating the block operator Δ_j on both sides, we have

$$\Delta_j\tilde{u}(t,x) = \int_0^t \Delta_j P_{s,t}\left(\mathscr{L}^{(\alpha)}_{\tilde{\kappa},\tilde{\sigma}} - \mathscr{L}^{(\alpha)}_{\tilde{\kappa}_0,\tilde{\sigma}_0}\right)\tilde{u}(s,x)\mathrm{d}s + \int_0^t \Delta_j P_{s,t}(\tilde{b}\cdot\nabla\tilde{u})(s,x)\mathrm{d}s + \int_0^t \Delta_j P_{s,t}\tilde{f}(s,x)\mathrm{d}s, \tag{5.2}$$

where

$$\int_0^t P_{s,t}f(s,x)\mathrm{d}s = \int_0^t\int_{\mathbb{R}^d} p^{\tilde{\kappa}_0,\tilde{\sigma}_0}_{s,t}(y)f(s,x+y)\mathrm{d}y\mathrm{d}s. \tag{5.3}$$

Convolution to Product

- Recall $\|u(t)\|_{\mathbf{B}^{\alpha}_{\infty,\infty}} = \sup_{j\geqslant 0} 2^{\alpha j}\|\Delta_j u(t)\|_\infty$.
- Noticing
$$|\Delta_j u(t,\theta_t)| = |\Delta_j \tilde{u}(t,0)|,$$
we get $\|\Delta_j u(t)\|_\infty$ by taking the supremum of the θ_t's initial data x_0 for $|\Delta_j \tilde{u}(t,0)|$.
- Therefore, we only need to consider values at the origin point:

$$\Delta_j \tilde{u}(t,0) = \int_0^t \Delta_j P_{s,t}\left(\mathscr{L}^{(\alpha)}_{\tilde{\kappa},\tilde{\sigma}} - \mathscr{L}^{(\alpha)}_{\tilde{\kappa}_0,\tilde{\sigma}_0}\right)\tilde{u}(s,0)\mathrm{d}s + \int_0^t \Delta_j P_{s,t}(\tilde{b}\cdot\nabla\tilde{u})(s,0)\mathrm{d}s + \int_0^t \Delta_j P_{s,t}\tilde{f}(s,0)\mathrm{d}s. \tag{5.4}$$

Here,

$$\int_0^t \Delta_j P_{s,t} f(s,0)\mathrm{d}s = \int_0^t \int_{\mathbb{R}^d} p^{\tilde{\kappa}_0,\tilde{\sigma}_0}_{s,t}(y)\cdot \Delta_j f(s,y)\mathrm{d}y\mathrm{d}s. \tag{5.5}$$

Convolution $\Longrightarrow$ Product

A crucial Lemma for heat kernels

Lemma 13 (Heat kernel estimates)

For any $T > 0$, $\beta \in [0, \alpha)$, *and* $n \in \mathbb{N}_0$, *there is a constant* $c = (d, m, c_0, \nu^{(\alpha)}, \beta, \alpha)$ *such that for any* $s, t \in [0, T]$ *and* $j \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} |x|^\beta |\Delta_j p_{s,t}^{\tilde{\kappa}_0, \tilde{\sigma}_0}(x)| \mathrm{d}x \leqslant c 2^{(m-n)j} (t-s)^{-\frac{n}{\alpha}} ((t-s)^{\frac{1}{\alpha}} + 2^{-j})^\beta, \tag{5.6}$$

and

$$\int_0^t \int_{\mathbb{R}^d} |x|^\beta |\Delta_j p_{s,t}^{\tilde{\kappa}_0, \tilde{\sigma}_0}(x)| \mathrm{d}x \mathrm{d}s \leqslant c 2^{-(\alpha+\beta)j}. \tag{5.7}$$

- **Note** We use the regularity of time to get the regularity of space.
- Under $(\mathbf{H}_\kappa^\beta)$, $(\mathbf{H}_\sigma^\beta)$ and $(\mathbf{H}_b^\beta)$, we have

$$\begin{cases} |\tilde{\kappa}(t,x,z) - \tilde{\kappa}_0(t,z)| \leqslant [\kappa(t,\cdot,z)]_{\mathbf{C}^\beta} |x|^\beta \leqslant c_0 |x|^\beta; \\ |\tilde{\sigma}(t,x) - \tilde{\sigma}_0(t)| \leqslant [\sigma(t,\cdot)]_{\mathbf{C}^\gamma} |x|^\gamma \leqslant c_0 |x|^\gamma; \\ |\tilde{b}(t,x)| \leqslant [b(t,\cdot)]_{\mathbf{C}^\beta} |x|^\beta \leqslant c_0 |x|^\beta. \end{cases} \tag{5.8}$$

Thank you for your attention!

Thank Zimo Hao for his advices to this presentation.